International Journal of Theoretical Physics, Vol. 34, No. 8, 1995

Spectral Decompositions of Operators on Non-Archimedean Orthomodular Spaces

Hans A. Keller[1] and Herminia Ochsenius A.[2]

Received March 28, 1995

The most central property of an infinite-dimensional Hilbert space is expressed by the projection theorem: Every orthogonally closed linear subspace is an orthogonal summand. Besides the obvious Hilbert spaces, there exist other infinite-dimensional orthomodular spaces. Here we study bounded linear operators on an orthomodular space E constructed over a field of generalized power series with real coefficients. Our main result states that every bounded, self-adjoint operator gives rise to a representation of E as the closure of an infinite orthogonal sum of invariant subspaces each of which is of dimension 1 or 2. The proof combines the technique of reduction modulo the residual spaces with theorems on orthogonal decompositions of finite matrices over fields of power series.

1. INTRODUCTION

Let E be a vector space over an involutory field $(K, *)$ endowed with a Hermitian form $\Phi: E \times E \to K$. We always assume that Φ is anisotropic, i.e., $\Phi(x, x) = 0$ only when $x = 0$. Then (E, Φ) is called orthomodular if every orthogonally closed linear subspace is a direct linear supplement of the whole space,

(P) $\quad U \subseteq E, U = U^{\perp\perp} \Rightarrow E = U \oplus U^{\perp}$

where $U^{\perp} := \{x \in E \mid \Phi(x, u) = 0 \text{ for all } u \in U\}$.

If dim $E < \infty$, then the projection theorem (P) is trivially true since Φ is anisotropic. In infinite dimension, however, condition (P) is a very strong one. For a long time it was an open problem whether there exist infinite-dimensional orthomodular spaces different from the classical Hilbert spaces.

[1]Technikum Luzern, CH-6048 How, Switzerland.
[2]Facultad de Matemáticas, Universidad Católica de Chile, Casilla 306, Coreo 22, Santiago de Chile, Chile.

0020-7748/95/0800-1507$07.50/0

Then, in 1979, a new class of such spaces was discovered (Gross and Künzi, 1980). All of them are constructed over non-Archimedeanly valued, complete fields and are endowed with a natural non-Archimedean norm with respect to which they are complete. Since they were first introduced, many aspects of these spaces, such as, for example, Clifford algebras, measures, and orthogonal groups, have been investigated.

In Keller and Ochsenius (1994, 1995a) we began a study of bounded linear operators on non-Archimedean orthomodular spaces. In spite of numerous analogies between such operators and their classical counterparts, there arise some striking differences which sharply contrast them with their counterparts on Hilbert spaces. For example, in Keller and Ochsenius (1994) we constructed an algebra of bounded, self-adjoint linear operators each of which has a one-point spectrum but does not admit any eigenvector, in fact not even an invariant subspace. These new features are rooted in the fact that the base field is never algebraically closed.

In the examples mentioned above the underlying field is the completion of a field of rational functions, so its arithmetic is utmost rigid. In the present paper we will concentrate on a space over a field $K = \mathbf{R}((\Gamma))$ of generalized power series with exponents in an additive group Γ of infinite rank and with real coefficients. Although this field is still far from being algebraically closed, its arithmetic is much smoother and the above strange phenomena nearly disappear. In fact, we shall reach a theorem on orthogonal decompositions of bounded operators which is surprisingly close to the classical spectral theorem.

To our non-Archimedean space (E, Φ) there is associated a sequence of residual spaces $\hat{E}_n$, $n = 0, 1, \ldots$, which turn out to be finite-dimensional inner product spaces over fields of power series in finitely many variables. A bounded operator T on E induces an operator $\hat{T}_n$ on each $\hat{E}_n$ (from some $n = n_0$ on), and an orthogonal decomposition of T automatically induces such a decomposition of every $\hat{T}_n$. The task of finding a spectral representation of T is thereby related to the problem of decomposing finite matrices over fields of power series. We begin by examining the matrix problem.

In Section 1 we present a basic result (Theorem 2), which states that if $K = \mathbf{R}((t_1, \ldots, t_m))$ is a field of power series with real coefficients, then every symmetric matrix can be orthogonally diagonalized over K. The proof is based on a recursive construction and aims at an effective computation of the transition matrix. In Section 3 we examine orthogonal decompositions of (nonsymmetric) matrices which are self-adjoint with respect to more general inner products. Finally, in Section 4, these results are combined with the technique of reduction onto the residual spaces in order to establish the main theorem on orthogonal decompositions of infinite-dimension, bounded, self-adjoint operators.

The paper is expository. We shall outline the crucial proofs in order to throw the underlying ideas into relief; formal details will be omitted.

2. DIAGONALIZATION OF SYMMETRIC MATRICES

Given any field K_0 with $char(K_0) \neq 2$, we let $K = K_0((t))$ be the field of formal power series in the indeterminate t with coefficients in K_0, and we let $v: K \to \mathbf{Z} \cup \{\infty\}$ be the usual exponential valuation. Thus for a typical $\alpha = \Sigma_{i \in \mathbf{Z}} a_i t^i$ in K we have $v(\alpha) = min\{i \in \mathbf{Z} \mid a_i \neq 0\}$ if $\alpha \neq 0$, $v(\alpha) = \infty$ if $\alpha = 0$. The valued field (K, v) is complete and henselian (Ribenboim, 1964; Schilling, 1950).

We consider the ring $Mat_n(K)$ of all square matrices of size $n \times n$ with entries in K along with the subring $Mat_n(K_0)$ consisting of all matrices with entries in the subfield $K_0 \subset K$. We shall denote the matrices in $Mat_n(K)$ by $\mathcal{A}, \ldots, \mathcal{U} \ldots$ and those in $Mat_n(K_0)$ by $A, \ldots, U, \ldots$. The unit matrix is always denoted by I.

A matrix $\mathcal{A} \in Mat_n(K)$ is called *orthogonal* if its transpose $\mathcal{A}^*$ is equal to the inverse $\mathcal{A}^{-1}$, i.e., if $\mathcal{A}^*\mathcal{A} = \mathcal{A}\mathcal{A}^* = I$. We say that $\mathcal{A}$ is *diagonal* if all entries outside the main diagonal are 0.

We can now state the main result of this section.

Theorem 1 (Keller and Ochsenius, 1995a). Let $K = K_0((t))$ and $n \geq 1$. The following conditions are equivalent:

(a) Every symmetric matrix $A \in Mat_n(K_0)$ can be diagonalized by means of an orthogonal matrix $U \in Mat_n(K_0)$.

(b) Every symmetric matrix $\mathcal{A} \in Mat_n(K)$ can be diagonalized by means of an orthogonal matrix $\mathcal{U} \in Mat_n(K)$.

Outline of the Proof. We only deal with the difficult part, namely the implication (a) $\Rightarrow$ (b). Its proof is divided into several steps.

1. Let there be given a symmetric matrix $\mathcal{A}$ of size $n \times n$ in $Mat_n(K)$. We may assume that all its entries α_{ij} satisfy $v(\alpha_{ij}) \geq 0$. Expanding each α_{ij} as a power series $\alpha_{ij} = a_{ij}^{(0)} + a_{ij}^{(1)} t + a_{ij}^{(2)} t^2 + \cdots + a_{ij}^{(m)} t^m + \cdots$ and collecting the coefficients of the same powers, we obtain a representation

$$\mathcal{A} = A_0 + A_1 \cdot t + A_2 \cdot t^2 + \cdots + A_m \cdot t^m + \cdots$$

where the A_m are symmetric matrices with entries in K_0.

2. It is easy to show that we may assume, without loss of generality, that the initial matrix A_0 is diagonal but not a multiple of the unit matrix. It is only in this preliminary step where the hypothesis (a) is actually used.

3. The orthogonal matrix $\mathcal{U}$ we are looking for will have the form

$$\mathcal{U} = U_0 + U_1 \cdot t + U_2 \cdot t^2 + \cdots + U_m \cdot t^m + \cdots \qquad (1)$$

where $U_m \in Mat_n(K_0)$. The idea is to construct recursively matrices U_0, U_1, . . . in $Mat_n(K_0)$ such that the resulting matrix $\mathcal{U}$ given by (1) satisfies both

$$\mathcal{U}^*\mathcal{U} = \mathcal{I}$$

and

$$\mathcal{U}^*\mathcal{A}\mathcal{U} \quad \text{is diagonal}$$

These two conditions hold if and only if

$$U_0^*U_0 = I, \qquad \sum_{i+j=m} U_i^*U_j = 0 \quad \text{for all} \quad m \geq 1 \tag{2}$$

and

$$\sum_{i+j+k=m} U_i^*A_jU_k \quad \text{is diagonal} \quad \text{for all } m \geq 0 \tag{3}$$

4. Now there arise two cases. If the diagonal entries of A_0, i.e., the eigenvalues of A_0, are pairwise different, then the recursive construction can be carried out directly starting with $U_0 := I$. The conditions (2), (3) are just what we need to compute U_m from $U_{m-1}, \ldots, U_0, A_m, \ldots, A_0$. However, this is not possible when A_0 has some eigenvalues repeated. The reason is that in the step from U_{m-1} to U_m we cannot compute all the entries of U_m; unavoidably we have to make choices which in turn will have a strong impact on the computations in the next steps. These difficulties are not unexpected. Indeed, if A_0 has some eigenvalues repeated, then the matrix $\mathcal{A}$ may have multiple eigenvalues and consequently the transition matrix $\mathcal{U}$ is not unique. In the general case of multiple eigenvalues we cannot diagonalize $\mathcal{A}$ at once. What can be done is to produce, by the above recursive construction, an orthogonal matrix $\mathcal{U}$ such that $\mathcal{U}^*\mathcal{A}\mathcal{U}$ is decomposed into two blocks. The proof is then completed by induction on the size of $\mathcal{A}$.

It is a remarkable feature of the above proof that it does not involve the spectrum of the symmetric matrix $\mathcal{A}$. The eigenvalues are in fact obtained as a by-product of the recursive construction.

It is well known that over the field **R** of real numbers every symmetric matrix can be diagonalized. Applying Theorem 1 repeatedly, we obtain the following result.

Theorem 2. Let $K = \mathbf{R}((t_1, \ldots, t_m))$ be a field of power series in finitely many variables and real coefficients. Then every symmetric matrix $\mathcal{A} \in Mat_n(K)$ can be diagonalized by means of an orthogonal matrix $\mathcal{U} \in Mat_n(K)$.

As an immediate consequence we have the following result.

Corollary. If $\mathcal{A}$ is a symmetric square matrix with entries in a field of power series $K = \mathbf{R}((t_1, \ldots, t_m))$, then its characteristic polynomial

$$p_{\mathscr{A}}(\lambda) = \det(\mathscr{A} - \lambda \cdot \mathscr{I})$$

decomposes completely into linear factors.

3. DECOMPOSITIONS OF SELF-ADJOINT OPERATORS

In order to proceed further we first transfer Theorem 2 to the geometric framework of vector spaces and linear operators.

Let K be one of the fields $K = \mathbf{R}((t_1, \ldots, t_m))$ and consider the vector space $E_n := K^{n+1}$ together with the canonical inner product $\langle \cdot, \cdot \rangle$ defined by

$$\langle (\xi_0, \ldots, \xi_n), (\eta_0, \ldots, \eta_n) \rangle = \sum_{i=0}^{n} \xi_i \eta_i$$

A linear operator T: $E_n \to E_n$ is self-adjoint, i.e., $\langle T(x), y \rangle = \langle x, T(y) \rangle$ for all $x, y \in E_n$, iff its matrix (with respect to the canonical base of E_n) is symmetric. Thus we may restate Theorem 2 as follows.

Theorem 2′. Assume that the linear operator T: $E_n \to E_n$ is self-adjoint with respect to the canonical inner product $\langle \cdot, \cdot \rangle$. Then E_n decomposes into an orthogonal direct sum of invariant subspaces of dimension 1.

As explained in the introduction, our purpose is to establish orthogonal decompositions of bounded, self-adjoint linear operators on an infinite-dimensional orthomodular space by examining their reductions to the finite-dimensional residual spaces. Theorem 2′ is not strong enough for that purpose. The reason is that the residual spaces are endowed with inner products (i.e., positive-definite, symmetric bilinear forms) which are essentially different from the canonical ones.

We therefore turn to finite-dimensional operators which are self-adjoint with respect to more general inner products. Specifically, we consider spaces (E_n, Φ_n), $n = 1, 2, \ldots$, of the following kind. The base field of (E_n, Φ_n) is the field of power series $K_n = \mathbf{R}((t_1, \ldots, t_n))$, E_n is an $(n + 1)$-dimensional vector space over K_n with base $\{e_0, \ldots, e_n\}$, and Φ_n: $E_n \times E_n \to K_n$ is the bilinear form given by

$$\Phi_n(e_i, e_j) = \Phi_n(e_j, e_i) = 0 \qquad \text{for} \quad 0 \le i < j \le n$$

$$\Phi_n(e_0, e_0) = 1; \qquad \Phi_n(e_i, e_i) = t_i \qquad \text{for} \quad 1 \le i \le n$$

We shall express this by writing $\Phi_n \simeq \mathrm{diag}(1, t_1, \ldots, t_n)$. Notice that Φ_n is positive definite with respect to the ordering on $\mathbf{R}((t_1, \ldots, t_n))$ by powers of $t_1, \ldots, t_n$.

We cannot expect that the conclusion of Theorem 2′ carries over to self-adjoint operators on (E_n, Φ_n), as can be seen by the following simple example.

Let $K_1 := \mathbf{R}((t_1))$ and consider the two-dimensional space (E_1, Φ_1) over K_1 where $\Phi_1 \simeq \mathrm{diag}(1, t_1)$. Then the operator $T\colon E_1 \rightarrow E_1$ given by the matrix

$$\begin{bmatrix} 0 & 1 \\ t_1 & 0 \end{bmatrix}$$

is certainly self-adjoint. However, T has no eigenvector because its characteristic polynomial is

$$p_t(\lambda) = \lambda^2 - t_1$$

which has no root in $\mathbf{R}((t_1))$.

Thus if $T\colon E_n \rightarrow E_n$ is self-adjoint with respect to $\Phi_n \simeq \mathrm{diag}(1, \ldots, t_n)$, then E_n contains, in general, invariant indecomposable subspaces of dimension 2. The question is whether there exist indecomposable subspaces of even higher dimension. The next result shows that this does not happen.

Theorem 3. Let E_n be an $(n + 1)$-dimensional vector space over $\mathbf{R}((t_1, \ldots, t_n))$ and assume that the linear operator $T\colon E_n \rightarrow E_n$ is self-adjoint with respect to the inner product $\Phi_n \simeq \mathrm{diag}(1, t_1, \ldots, t_n)$. Then E_n decomposes into an orthogonal direct sum of invariant subspaces each of which is of dimension 1 or 2.

The proof relies on the central idea used to establish Theorem 1: The orthogonal matrix $\mathcal{U} = U_0 + U_1 t + U_2 t^2 \ldots$ which decomposes the matrix of T is generated by a recursive procedure. The general step in the recursive construction is considerably more involved and requires a careful subdivision of matrices into several regions.

4. OPERATORS ON INFINITE-DIMENSIONAL ORTHOMODULAR SPACES

In this final section we study self-adjoint operators on an infinite-dimensional orthomodular space over a field of generalized power series. The space in question can be obtained from the spaces (E_n, Φ_n) (cf. Section 3) by first taking inductive limits, then extending the field of scalars to its maximal completion and then completing the space. We will not carry out the details, but will give an explicit description of the space thus constructed.

4.1. The Base Field

We start with a direct sum

$$\Gamma := \mathbf{Z} \oplus \mathbf{Z} \oplus \cdots \oplus \mathbf{Z} \oplus \cdots$$

of countably many copies of the group of integers. Γ is an Abelian, additive group under componentwise operations. We order Γ antilexicographically.

Let $K := \mathbf{R}((\Gamma))$ be the field of all generalized power series with exponents in Γ and coefficients in $\mathbf{R}$. Thus K consists of all series

$$\xi = \sum_{\gamma \in \Gamma} a_\gamma t^\gamma \qquad (a_\gamma \in \mathbf{R})$$

for which the support

$$\operatorname{supp}(\xi) := \{\gamma \in \Gamma \,|\, a_\gamma \neq 0\}$$

is a well-ordered subset of Γ. The operations on K are the obvious ones: if $\xi = \Sigma_{\gamma \in \Gamma}\, a_\gamma t^\gamma$ and $\eta = \Sigma_{\gamma \in \Gamma}\, b_\gamma t^\gamma$, then $\xi + \eta = \Sigma_{\gamma \in \Gamma}\, (a_\gamma + b_\gamma) t^\gamma$ and $\xi \cdot \eta = \Sigma_{\gamma \in \Gamma}\, c_\gamma t^\gamma$, where $c_\gamma := \Sigma_{\delta + \delta' = \gamma}\, a_\delta b_{\delta'}$.

There is a natural Krull valuation v: $K \to \Gamma \cup \{\infty\}$ on K defined by

$$v(\xi) := \min \operatorname{supp}(\xi) \quad \text{if } \xi \neq 0; \qquad v(\xi) = \infty \quad \text{if } \xi = 0.$$

The valued field (K, v) is a complete and henselian (Ribenboim, 1964).

For $i \in \mathbf{N}$ we let

$$\gamma_i := (0, \ldots, 0, 1, 0, 0, \ldots) \in \Gamma \qquad \text{where 1 is in the } i\text{th place}$$

and we let t_i be the one-term series $t_i := 1 \cdot t^{\gamma_i}$. Then the topological closure of the subfield of (K, v) generated by $\{t_1, t_1, \ldots, t_n\}$ is (isomorphic to) $\mathbf{R}((t_1, \ldots, t_n))$. Thus (K, v) contains an isomorphic copy of each of the fields K_n considered in Section 3.

4.2. The Space (E, Φ)

For simplicity we put $t_0 := 1$. The set

$$E := \left\{ (\xi_i)_{i \in \mathbf{N}_0} \in K^{N_0} \,\middle|\, \text{the series } \sum_{i=0}^{\infty} \xi_i^2 t_i \text{ converges in the valuation topology} \right\}$$

is a vector space under componentwise operations. We define a symmetric, bilinear form Φ on E by

$$\Phi(x, y) := \sum_{i=0}^{\infty} \xi_i \eta_i t_i \qquad \text{for} \quad x = (\xi_i)_{i \in \mathbf{N}_0}, \quad y = (\eta_i)_{i \in \mathbf{N}_0}$$

This completes the construction of the space (E, Φ).

4.3. Basic Properties of (E, Φ)

The most important property is the following.

Theorem 4 (Gross and Künzi 1980, Theorem 28). (E, Φ) is an orthomodular space.

It is readily verified that the map $\|\cdot\|: E \to \Gamma \cup \{\infty\}$ defined by

$$\|x\| := v(\Phi(x, x))$$

is a non-Archimedean norm on E. The norm-topology, defined by taking the sets $\{x \in E \mid \|x\| \geq \gamma\}$ (where γ varies over Γ) as a zero-neighborhood basis, turns E into a topological vector space. The form Φ is continuous in the norm topology.

Theorem 5 (Gross and Künzi, 1980, Theorem 28). (i) E is complete in the norm topology, i.e., a Banach space.

(ii) A linear subspace U of E is closed in the norm topology if and only if it is orthogonally closed.

4.4. The Standard Base

For $i \in \mathbf{N_0}$ we let

$$e_i := (0, \ldots, 0, 1, 0, \ldots) \in E$$

be the vector that has 1 in place $(i + 1)$ and 0 in all other places. Then $\Phi(e_i, e_j) = 0$ for $i \neq j$ and $\Phi(e_i, e_i) = t_i$. Now, $\{e_i \mid i \in \mathbf{N_0}\}$ is an orthogonal continuous base of (E, Φ), that is, every vector $x \in E$ can be expressed as

$$x = \sum_{i=0}^{\infty} \xi_i e_i = \lim_{n \to \infty} \left(\sum_{i=0}^{n} \xi_i e_i \right)$$

The space (E, Φ) contains an isometric copy of each of the spaces (E_n, Φ_n) considered in Section 3, namely the subspace generated by $\{e_0, \ldots, e_n\}$ over $\mathbf{R}((t_1, \ldots, t_n)) \subset K$.

4.5. Residual Spaces

For $n = 0, 1, 2, \ldots$ the set

$$\Delta_n := \underbrace{\mathbf{Z} \oplus \cdots \oplus \mathbf{Z}}_{n \text{ times}} \oplus \{0\} \oplus \{0\} \oplus \cdots \subset \Gamma$$

is a convex (or isolated) subgroup of Γ. To each Δ_n there corresponds, by general valuation theory, a valuation ring $R_n := \{\xi \in K \mid \varphi(\xi) \geq \delta$ for some $\delta \in \Delta_n\}$ with maximal ideal J_n and residue field $\hat{K}_n := R_n/J_n$. The sets

$$M_n := \{x \in E \mid \Phi(x, x) \in R_n\}, \qquad S_n := \{x \in E \mid \Phi(x, x) \in J_n\}$$

of all "medium" and all "infinitely small" vectors are R_n-submodules of E. The quotient $\hat{E}_n := M_n/S_n$ is naturally a vector space over $\hat{K}_n$. Moreover, the form Φ induces a symmetric bilinear form $\hat{\Phi}_n$ on $\hat{E}_n$. We call $(\hat{E}_n, \hat{\Phi}_n)$ the residual space of (E, Φ) belonging to the convex group Δ_n.

It is readily verified that:

(i) $\hat{K}_n$ is isomorphic to $\mathbf{R}((t_1, \ldots, t_n))$.
(ii) $(\hat{E}_n, \hat{\Phi}_n)$ is isometric to (E_n, Φ_n).

Let π_n: $M_n \to \hat{E}_n = M_n/S_n$ be the canonical epimorphism. Every linear subspace U of E is reduced under π_n to a linear subspace $\hat{U}_n = \pi_n(U) := \{\pi_n(x) \mid x \in U \cap M_n\}$. The reduction map π_n preserves orthogonality, i.e., if $U \perp W$, then $\pi_n(U) \perp \pi_n(W)$.

4.6. Bounded Linear Operators

A linear operator T: $E \to E$ is called bounded if the subset

$$\{\|T(x)\| - \|x\| \mid 0 \neq x \in E\} \subset \Gamma$$

has an upper bound in Γ. Clearly, a bounded operator T is continuous with respect to the norm topology, and consequently T is determined by its action on the vectors e_i of the standard base. It follows that a bounded operator can be represented by a countably infinite matrix.

We observe that, in contrast to the classical case of Hilbert spaces, there exist linear operators on (E, Φ) which are continuous but not bounded. Notice also that a bounded operator cannot be assigned a norm in the usual way because a bounded subset of Γ may fail to have a supremum.

4.7. The Main Result

We now turn to decompositions of operators.

Theorem 6. Let (E, Φ) be the orthomodular space constructed in Sections 4.1, 4.2. Then every bounded, self-adjoint linear operator T: $E \to E$ gives rise to a representation of E as the closure of an orthogonal direct sum of countably many invariant subspaces each of which is of dimension 1 or 2. Thus T can be represented as

$$T = \sum_{i=0}^{\infty} Q_i$$

where the Q_i are pairwise orthogonal operators of rank 1 or 2.

Outline of the Proof. 1. Let there be given a bounded, self-adjoint operator T: $E \to E$. Multiplying T by some scalar, we may assume that T is bounded by $\gamma = 0$; thus $\|T(x)\| - \|x\| \geq 0$ for all $0 \neq x \in E$. Then T induces an operator $\hat{T}_n$: $\hat{E}_n \to \hat{E}_n$ on each residual space $\hat{E}_n$, $n = 0, 1, \ldots$. We shall identify $(\hat{E}_n, \hat{\Phi}_n)$ with its isometric copy (E_n, Φ_n) in (E, Φ) and accordingly we write T_n instead of $\hat{T}_n$.

2. Clearly T_n is self-adjoint with respect to $\Phi_n \sim \mathrm{diag}(1, \ldots, t_n)$. Hence, by Theorem 3, every E_n can be written as an orthogonal sum

$$E_n = L_n^{(0)} \oplus^\perp \cdots \oplus^\perp L_n^{(q_n)} \oplus^\perp P_n^{(0)} \oplus^\perp \cdots \oplus^\perp P_n^{(r_n)} \tag{4}$$

of invariant straight lines $L_n^{(i)}$ and invariant planes $P_n^{(j)}$. The task is to show that these decompositions, when properly chosen, can be lifted to the infinite-dimensional space E. To this end we have to interrelate them by means of the reduction maps π_n.

3. Recall that π_n preserves orthogonality. Moreover, if a subspace $U_{n+1} \subseteq E_{n+1}$ is invariant under T_{n+1}, then $\pi_n(U_{n+1}) \subseteq E_n$ is invariant under T_n. Notice that either $\dim \pi_n(U_{n+1}) = \dim U_{n+1}$ or $\dim \pi_n(U_{n+1}) = \dim U_{n+1} - 1$.

4. Let us consider first the very special case where in every decomposition (4) there occur only straight lines and, additionally, these straight lines correspond to pairwise different eigenvalues $\lambda_n^{(i)}$ $(0 \le i \le n)$ of T_n. Thus

$$E_n = L_n^{(0)} \oplus^\perp \cdots \oplus^\perp L_n^{(n+1)} \tag{5}$$

The assumptions entail that the decompositions (5) are unique. Applying π_n to $E_{n+1} = L_{n+1}^{(0)} \oplus^\perp \cdots \oplus^\perp L_{n+1}^{(n+1)}$, we get

$$\pi_n(E_{n+1}) = E_n = \pi_n(L_{n+1}^{(0)}) \oplus^\perp \cdots \oplus^\perp \pi_n(L_n^{(n+1)}) = L_n^{(0)} \oplus^\perp \cdots \oplus^\perp L_n^{(n)}$$

By uniqueness we conclude that, after renumbering,

$$\pi_n(L_{n+1}^{(i)}) = L_n^{(i)} \quad \text{for } 0 \le i \le n, \qquad \pi_n(L_{n+1}^{(n+1)}) = \{0\}$$

Hence we can pick eigenvectors $f_n^{(i)} \in L_n^{(i)}$ such that

$$\pi_n(f_{n+1}^{(i)}) = f_n^{(i)} \qquad \text{for all} \quad i \in \mathbf{N} \text{ and all } n \ge i$$

Then $(f_n^{(i)})_{n \ge i}$ is a Cauchy sequence. Let $g^{(i)} := \lim_{n\to\infty} f_n^{(i)}$. It is readily verified that $g^{(i)}$ is an eigenvector for T and $g^{(i)} \perp g^{(j)}$ for $i \ne j$. The orthogonal family $\{g^{(i)} \mid i \in \mathbf{N}\}$ is maximal in (E, Φ) and so it provides the required decomposition of the operator T in the present special case.

5. The general case is much more involved, by lack of uniqueness. The first step consists in proving that in any representation (4) the invariant, indecomposable planes $P_n^{(j)}$ are uniquely determined by the operator T_n; the proof of this fact makes essential use of the arithmetic properties of the base field $\mathbf{R}((t_1, \ldots, t_n))$ as well as the geometry of the space (E_n, Φ_n). Next, collecting in (4) all straight lines that correspond to the same eigenvalue, we obtain a decomposition

$$E_n = U_n^{(0)} \oplus^\perp \cdots \oplus^\perp U_n^{(s_n)} \oplus^\perp P_n^{(0)} \oplus^\perp \cdots \oplus^\perp P_n^{(r_n)} \tag{6}$$

where each $U_n^{(i)}$ is the eigenspace of some eigenvalue $\lambda_n^{(i)}$ of T_n.

Now the decompositions (6) are unique. We observe that an indecomposable plane $P_{n+1}^{(j)} \subseteq E_{n+1}$ is reduced under π_n to either an indecomposable plane or an invariant straight line of E_n. Furthermore, $\pi_n(U_{n+1}^{(j)})$ is spanned by eigenvectors belonging to the same eigenvalue of T_n. Notice, however, that two eigenspaces $U_{n+1}^{(i)}$ and $U_{n+1}^{(j)}$ belonging to different eigenvalues of T_{n+1} are possibly reduced to spaces $\pi_n(U_{n+1}^{(i)})$ and $\pi_n(U_{n+1}^{(j)})$ belonging to the same eigenvalue of T_n. From these remarks we deduce that every $P_n^{(i)}$ is equal to some $\pi_n(P_{n+1}^{(j)})$ and every $U_n^{(i)}$ is the orthogonal sum of some $\pi_n(U_{n+1}^{(j)})$'s.

It is convenient to add the zero-space $Z_n := \{0\} \subset E_n$ to every decomposition (6). Put

$$\mathscr{C} := \{U_n^{(i)}, P_n^{(j)}, Z_n | n \in \mathbf{N}_0, 0 \leq i \leq s_n, 0 \leq j \leq r_n\}$$

We define a partial ordering $<$ on $\mathscr{C}$ by the rules

$$U_n^{(i)} < U_{n+1}^{(j)} \Leftrightarrow \{0\} \neq \pi_n(U_{n+1}^{(j)}) \subseteq U_n^{(i)}; \qquad P_n^{(i)} < P_{n+1}^{(j)} \Leftrightarrow \pi_n(P_{n+1}^{(j)}) = P_n^{(i)}$$

$$U_n^{(i)} < P_{n+1}^{(j)} \Leftrightarrow \pi_n(P_{n+1}^{(j)}) \subseteq U_n^{(i)} \qquad Z_n < U_{n+1}^{(j)} \Leftrightarrow \pi_n(U_{n+1}^{(j)}) = Z_n$$

Then $(\mathscr{C}, <)$ is a tree. A combinatorial argument shows that the decompositions (6) can be refined in such a way that the resulting tree has no ramifications except on top of the zero-spaces Z_n. In each branch of this tree we can now proceed as in step 4. This completes the proof.

We have thereby reached a decomposition theorem which is remarkably close to the classical spectral theorem for operators on complex Hilbert spaces. The methods of proof, however, are entirely different.

ACKNOWLEDGMENT

The authors have been partially supported by FONDECYT, Proyecto No. 1930513.

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